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Extension of Maxwell's equations for non-stationary magnetic fluids using gauss's divergence theorem

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ABSTRACT

The work presented in this paper focuses on formulating the development of time-dependent electromagnetic field laws through the application of Gauss's divergence theorem. The first part of the discussion looks at the basic ideas of electromagnetism. It focuses on how classical formulations of the laws of electromagnetism can be adapted to account for non-stationary conditions, especially regarding magnetic fluids that don't conduct electricity. It is suggested that employing Gauss's divergence theorem could help improve the computational analysis of these generalized equations, which would make them more useful in magnetic fluid dynamics. The paper examines the intricate interactions between non-conductive particles and conductive fluids under magnetic fields. By putting these interactions into a single theoretical framework, this work aims to help us understand non-stationary electromagnetic phenomena and how they affect many different scientific and engineering fields. The concluding section of the study examines the prospective practical applications of these extended equations. They could enable the development of more advanced electromagnetic devices and systems. Creating a strong set of analytical tools that can find new scientific paths and useful applications is the main goal of the study, particularly in the areas of electromagnetic induction and fluid dynamics. This research offers potential for substantial progress in both theoretical comprehension and technological advancement, The proposed method is applicable to real-world systems such as ferrofluid-based cooling, magnetic dampers, plasma generators, and smart electromagnetic devices. These applications demonstrate the practical benefits of coupling field behavior with boundary dynamics using Gauss's theorem.

1. Introduction

Maxwell's equations serve as the cornerstone of classical electromagnetism, encapsulating the behavior of electric and magnetic fields through a coherent framework. These equations can be expressed in both vector and scalar forms, which play a pivotal role in understanding how electric charges and currents generate electric and magnetic fields. Gauss's law for electric fields and Gauss's law for magnetic fields illustrate the relationship between charge and the resulting field configurations [1,2]. Such principles highlight the importance of the enclosed charge in determining the electric field, which is foundational to the study of electromagnetic theory.

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Furthermore, the application of Gauss's theorem elucidates the processes of magnetic and electrical induction [3–5], extending our understanding beyond static fields to dynamic interactions. The vector forms of Maxwell's equations incorporate this theorem to provide comprehensive insights into how changing electric fields induce magnetic fields, aligning with the established criteria for electromagnetic behavior [6–8]. The substitution of these criteria gives rise to new formulations, expanding the applicability of Maxwell's equations in various physical contexts.

Moreover, the scalar form of these equations, particularly through Faraday's law of induction, emphasizes the significance of electric and magnetic fields' interactions [9–11]. The scalar potential field's steepest decline is intrinsically linked to induced electromotive forces, illustrating the foundational principles that govern electrical induction. Thus, Maxwell's equations, through their vector and scalar forms, encapsulate the dynamic interplay between electric and magnetic phenomena, firmly grounded in Gauss's theorem [12–13].

The study of non-conductive magnetic liquids in the presence of magnetic fields has garnered significant attention in recent years, as it presents unique challenges and opportunities for research [6,14,15]. Understanding the interactions between non-magnetic particles and conductive fluids is crucial, particularly in the context of strong magnetic fields. According to Sun et al. in [16], the interaction behaviors of non-magnetic particles within a conductive medium reveal intricate dynamics influenced by magnetic forces. This study fundamentally contributes to our knowledge of how such particles migrate and interact under varying conditions, emphasizing the importance of a detailed force analysis in such environments.

The review by Vadde et al. in [17] expands on the methods employed to characterize the flow of incompressible Newtonian liquids when subjected to magnetic and electric fields. Their findings highlight the complexities involved in non-invasive techniques, which are essential for observing the behavior of non-conductive fluids [18,19]. These insights affirm that various factors significantly influence the flow dynamics within non-conductive magnetic liquid systems, thereby establishing a framework for further investigations in this field.

Additionally, the research conducted by Sun et al. in [20] elucidates the solid-liquid and particle-particle interactions during processing, particularly emphasizing the role of micrometer-sized non-conductive particles in a conducting liquid. This work illustrates the multifaceted nature of magnetic field effects and the electrohydro-dynamic interactions that ensue, ultimately enhancing the understanding of non-conductive magnetic fluids in real-world applications. Collectively, these studies provide a comprehensive overview of the behavior of non-conductive magnetic liquids in magnetic fields, underscoring the significance of continued exploration in this promising area of research.

Many researchers have increasingly relied on Maxwell's equations to analyze and simulate complex behaviors in magnetically influenced fluid systems. Their work spans a wide range of applications, including MR dampers, electromagnetic welding, plasma generators, and ferrofluid dynamics. This growing trend reflects the versatility and effectiveness of Maxwell's framework in coupling electromagnetic theory with fluid mechanics, leading to enhanced modeling accuracy and deeper insight into electro-magneto fluidic phenomena. For instance, as addressed by both Wael Elsaady et al. in [21], Thierry Tchoumi et al. in [22], Sangeeta B. Punjabi et al. in [23], Fabian Klemens et al. in [24], Katja Henjes in [25], M. Lohakan et al. in [26], Peter Vadasz in [27], and P.H.N. Pimenta et al. in [28].

In light of the foregoing, this paper primarily aims to establish a novel expression of the time-dependent electromagnetic field laws based on the integral form of Gauss's law for divergence. This approach seeks to simplify the analysis of non-stationary magnetic fields and improve numerical solutions to these equations. The paper also aims to extend the application of this approach to magnetic fluid dynamics, providing a unified framework for studying interactions between magnetic fields and fluids. Through this new formulation, the paper seeks to open new horizons in understanding and analyzing non-stationary electromagnetic phenomena in a variety of scientific and engineering applications.

2. Fundamental Maxwell's equations in electromagnetism

2.1. Maxwell-gauss equation

Gauss's theorem is also derived from Coulomb's law. [29] Electric flux a closed surface S is equals the algebraic sum of the total charges Q_1 enclosed within the surface, multiplied by the factor $1/\varepsilon_0$ [30]:

$$\iint \bigcirc_{(s)} \vec{E}.\vec{ds} = \frac{1}{\varepsilon_0}\sum_1 Q_1 = \frac{1}{\varepsilon_0}\int\int_{(\tau)}\int \rho . d\tau \tag{1}$$

According to Ostrogradsky theorem, it is possible to write:

$$\iint \bigcirc_{(s)} \vec{E}.\vec{ds} = \int\int_{(\tau)}\int div\vec{E}\, d\tau \tag{2}$$

Where: τ is the volume bounded by surface S.From Eqs. (1) and (2), we can establish that:

$$div\vec{E} = \frac{\rho}{\varepsilon_0} \tag{3}$$

Ø Case of charges outside a closed surface (S)

The elements dS_1 and dS_2 are seen under the same angle $d\Omega$ in absolute value. However, $\overrightarrow{E_1}$ and $d\overrightarrow{S_1}$ are collinear while $\overrightarrow{E_2}$ and $d\overrightarrow{S_2}$ are opposite Fig. 1. The fluxes [31,32]:

$$\begin{cases} d\Phi_1 = \overrightarrow{E_1}.\overrightarrow{dS_1} \\ d\Phi_2 = \overrightarrow{E_2}.\overrightarrow{dS_2} \end{cases} \tag{4}$$

are therefore of opposite signs. Elementary fluxes cancel out in pairs, so the total flux $\overrightarrow{E_1}$ of the field created by charge Q outside the closed surface is zero.

Ø Scenario of charges within a confined surface (S)

Charges inside a sealed surface S represent the sum of the electric charge contained by a sealed surface. Since all surface element vectors point outward (Fig. 2), the net flux through S is nonzero and equals the sum of all elementary fluxes from the enclosed charge Q [13–15]:

$$\Phi = \frac{Q}{4\pi\varepsilon_0}.\int d\Omega \tag{5}$$

The unit of solid angle is defined as the angle that cuts out a unit area on a sphere of unit radius. Since the surface area of a unit sphere is ($4\pi R^2 = 4\pi$), the solid angle that encompasses the entire space from a single point has a value of 4π. When this sum is extended to all of space, we get $\Phi = Q/\varepsilon_0$. If there are n charges ($Q_1, Q_2 \ldots\ldots Q_n$) inside the surface S:

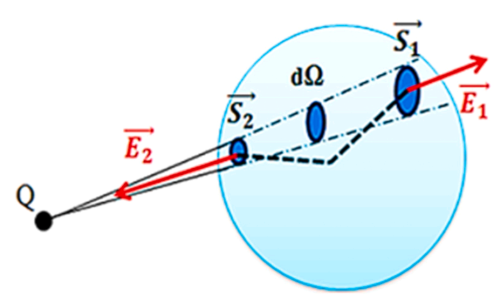

Fig. 1. Charges external to a closed surface S.

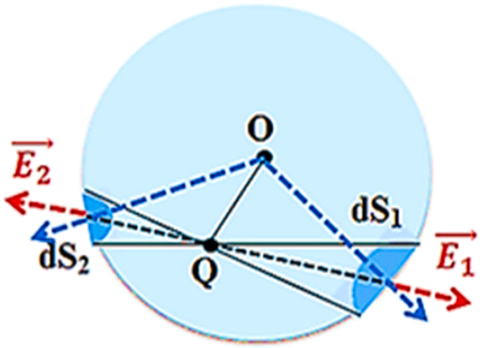

Fig. 2. Charges inside a closed surface S.

$$\Phi = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} Q_i \tag{6}$$

We impose that:$\Phi = \frac{Q_i}{\varepsilon_0}$

No matter what charges are on the outside, the flow of E through a closed surface is always equal to the sum of the charges inside divided by ε_0. We can show that it is just the local form of Gauss's theorem by using Eq. (3): We will integrate this equation over a volume τ that a surface S surrounds:

$$\iiint_{(\tau)} div\,\vec{E}\,d\tau = \frac{1}{\varepsilon_0}\iiint_{(\tau)} \rho.d\tau \rightarrow \oiint_{(s)} \vec{E}.\vec{ds} = \frac{1}{\varepsilon_0}Q_{int} \tag{7}$$

Q_{int} being the charges inside the surface S

2.2. Maxwell-Ampere equation

In magnetostatics, Ampère's theorem relates magnetic fields to electric currents and applies best in highly symmetrical cases [33,34].

The evolution of the concept from static to dynamic charges leads to the phenomenon of magnetism. Moving charges produce steady currents while maintaining charge distribution in space. This shift represents a step towards a more comprehensive understanding of electromagnetic phenomena ($\frac{\partial \rho}{\partial t} = 0$).

Current lines in magneto-statics must form closed loops. As a result, systems resembling capacitors, which would cause charge accumulation, are not permitted in magneto-static theory. This principle emphasizes the continuous nature of magnetic fields, contrasting with electrostatic systems where charge can accumulate ($\frac{\partial \rho}{\partial t} \neq 0 \rightarrow$ *Current not closed*). In this case, the charge conservation equation [33,34]:

$$\begin{cases} div\,\vec{J} = 0 \\ \nabla.J = 0 \end{cases} \tag{8}$$

The behavior of magnetic fields from steady current distributions is described by two fundamental principles from Maxwell's equations, forming the mathematical basis for understanding static magnetic phenomena [2,13,18]:

$$\begin{cases} div\,\vec{B} = 0 \\ \overrightarrow{rotB} = \mu_0\,\vec{J} \end{cases} \tag{9}$$

Eq. (9) indicates that the magnetic field is flux-conservative, signifying that the flux through any closed surface S equals zero:

$$\oiint_{(s)} \vec{B}.\vec{ds} = 0 \tag{10}$$

This signifies that the magnetic field lines neither converge nor diverge from any sources of magnetic fields. In other words, there are no magnetic charges comparable to electric charges that produce the electric field. Consequently, magnetic field lines must create closed loops. Consequently, Eq. (9) enables us to express [11]:

$$div\left(\overrightarrow{rotB}\right) = \mu_0 div\,\vec{J} = 0 \rightarrow div\,\vec{J} = 0 \tag{11}$$

This confirms that at any point in space, there can be no accumulation or spontaneous disappearance of charge. Using Eq. (9), we can write the following: let S be a surface resting on a contour C

$$\overrightarrow{rotB} = \mu_0\,\vec{J} \Rightarrow \iint_{(S)} \overrightarrow{rotB}\,\overrightarrow{dS} = \mu_0 \iint_{(S)} \vec{J}\,\overrightarrow{dS} \Rightarrow \oint_{(C)} \vec{B}.\vec{dl} = \mu_0.I \tag{12}$$

Consider the setup in Fig. 3, where an emf source is suddenly connected to condenser with plates in parallel, causing a time-dependent current J to flow through the wire. If Ampère's law is applied to loop C at a moment before the capacitor is fully charged (i.e., $J\neq0$), surface S_1 encloses a nonzero current, while surface S_2, which passes between the capacitor plates, encloses zero current since no conduction current flows through it. This discrepancy reveals that Ampère's law in its original form fails in this scenario, highlighting an inconsistency that necessitates a modification to the law itself [34].

Ampère's law in its standard form is incompatible here, requiring a modification to the theory.

We must therefore modify Ampère's law. To achieve this, Maxwell proposed adding an extra term, known as the displacement current Jd, to the real current J. Thus, Ampère's relation is revised accordingly, and we write:

$$\overrightarrow{rotB} = \mu_0\left[\vec{J} + \vec{J_d}\right] \tag{13}$$

Where $\vec{J_d}$ is an additional term that has the same physical dimension as a current density. This term is known as the displacement current. Let us now determine its expression. To do so, we begin by evaluating $div\,\vec{J}$.

$$\vec{J} = \frac{1}{\mu_0}\overrightarrow{rotB} - \vec{J_d} \tag{14}$$

Where:

$$div\,\vec{J} = div\left[\overrightarrow{rot\left(\frac{B}{\mu_0}\right)} - \vec{J_d}\right] = -div\vec{J_d} \tag{15}$$

However, the conservation of charge imposes $div\,\vec{J} = -\frac{\partial \rho}{\partial t}$ which implies:

$$div\vec{J_d} = \frac{\partial \rho}{\partial t} \tag{16}$$

If we consider that the Maxwell-Gauss relation remains valid in variable regime, then $\rho = \varepsilon_0 div\,\vec{E}$, and we arrive at the relation:

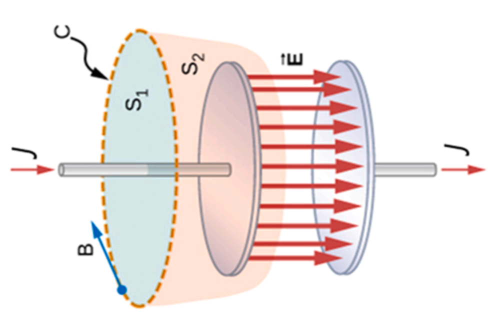

Fig. 3. discharge of the capacitor. Although S_1 and S_2 share the same boundary loop C, the currents flowing through them are different.

$$div\overrightarrow{J_d} = div\left(\varepsilon_0.\frac{\partial \overrightarrow{E}}{\partial t}\right) \tag{17}$$

where:

$$\overrightarrow{J_d} = \varepsilon_0\frac{\partial \overrightarrow{E}}{\partial t} \tag{18}$$

Lastly, the Maxwell-Ampère relation is presented in its final form as follows:

$$\overrightarrow{rotB} = \mu_0\left[\overrightarrow{J} + \varepsilon_0\frac{\partial \overrightarrow{E}}{\partial t}\right] \tag{19}$$

2.3. Maxwell-Faraday equation

Examine a wire loop C that encircles a surface S as depicted in Fig. 4. Around the circuit is a magnetic field B. If the magnetic flux Φ changes across this surface, it causes an electric current and, as a result, an electromotive force "e" in the circuit [17,20,35]:

$$e = -\frac{d\Phi}{dt} \tag{20}$$

The current arises from the field $d\overrightarrow{E}$. allowing us to write [36]:

$$e = \oint_{(C)} \overrightarrow{E}.\overrightarrow{dl} \tag{21}$$

The magnetic flux Φ of field $\overrightarrow{B}$ through surface S of the circuit is given by:

$$\Phi = \iint_{(S)} \overrightarrow{B}\,\overrightarrow{dS} \Rightarrow -\iint_{(S)} \frac{\partial \overrightarrow{B}}{\partial t}.\overrightarrow{dS} = \oint_{(C)} \overrightarrow{E}.\overrightarrow{dl} \tag{22}$$

And considering Stokes theorem [37]:

$$\iint_{(S)} \left[-\frac{\partial \overrightarrow{B}}{\partial t}.\overrightarrow{dS}\right] = \iint_{(S)} \overrightarrow{rotE}\,\overrightarrow{dS} \tag{23}$$

This results in the differential formulation of Faraday's law:

$$\overrightarrow{rotE} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{24}$$

We now show that Faraday's law in Eq. (20) is equivalent to Eq. (24). Using Fig. 3, we write:

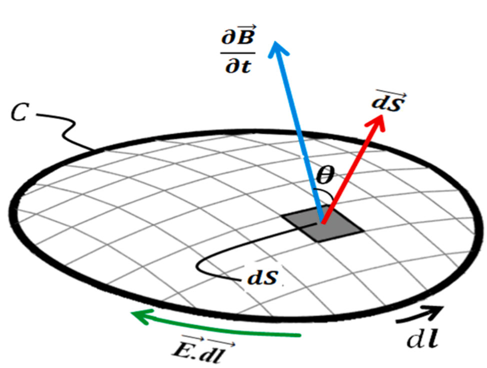

Fig. 4. Principle of Fraday Law.

$$\iint_{(S)} \overrightarrow{rotE}\,\overrightarrow{dS} = -\iint_{(S)} \frac{\partial \overrightarrow{B}}{\partial t}\overrightarrow{dS} \tag{25}$$

On the other hand, we have:

$$e = -\frac{d\Phi}{dt} = \oint_{(C)} \overrightarrow{E}.\overrightarrow{dl} = -\frac{\partial}{\partial t}\iint_{(S)} \overrightarrow{B}\,\overrightarrow{dS} \tag{26}$$

The Maxwell-Faraday Eq. (24) delineates the phenomenon of induction, wherein a fluctuating magnetic field generates an electric field [30].

3. Maxwell's equations in magnetic fluids

Magnetic and magnetorheological fluids, whose properties change under magnetic fields, are increasingly used. Their behavior is modeled using Maxwell's and Navier-Stokes equations, typically solved numerically [32,35]. The symmetry of operators in these models affects solution behavior. An example of Maxwell's equations in operator form follows:

$$\left\| \begin{matrix} -\frac{\partial}{\partial t} & rot \\ div & 0 \end{matrix} \right\| . \begin{pmatrix} D \\ H \end{pmatrix} = \begin{pmatrix} J \\ \rho \end{pmatrix} \tag{27}$$

$$\left\| \begin{matrix} \frac{\partial}{\partial t} & rot \\ div & 0 \end{matrix} \right\| . \begin{pmatrix} B \\ E \end{pmatrix} = 0 \tag{28}$$

In a nonconductive environment, the symmetry of the problem is evident (*j*=*ρ*=*0*). Non-stationary fields exhibit rotational characteristics (*E*=*-gradV*), requiring novel solution methodologies for non-stationary issues. Operator equations in summation notation further elucidate this concept:

$$(rotE)_i = \varepsilon_{ijk}.\frac{\partial E_k}{\partial x_j} \tag{29}$$

This expression employs standard derivatives rather than covariant derivatives. The disparity persists identically for both categories. The expression constitutes an antisymmetric matrix, which in three dimensions corresponds to a vector. We use the Levi-Civita tensor ε to make the dual vector of a second-order tensor that is not symmetric. The curl of a three-dimensional vector field is defined as the dual of the rotational tensor [38,39]:

$$(rotE)_i = \varepsilon_{ijk}.\frac{1}{2}.\left[\frac{\partial E_k}{\partial x_j} - \frac{\partial E_j}{\partial x_k}\right] \tag{30}$$

Gauss's divergence theorem reveals that any spatial change in a variable *g(x,t)* corresponds to a boundary response. Changes at S affect *V*, and vice versa. Fig. 5 illustrates this. The theorem states [15]:

$$\int_V \frac{\partial g(x,t)}{\partial x_i} dV = \int_S g(x,t)n_i.dS \tag{31}$$

This relationship is crucial for understanding the interplay between volume and boundary effects in physical systems.

This principle can be applied in qualitative analysis of electromagnetic fields, development of numerical methods for solving related equations, and optimization of electromagnetic fields. Its main advantage is providing precise insights into boundary conditions' effects on the problem, both qualitatively and quantitatively. The principle's importance is heightened because determining boundary conditions is often the researcher's responsibility, making understanding their impact crucial in scientific [37,40,41].

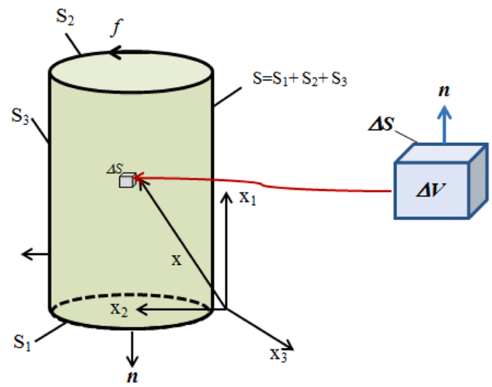

Fig. 5. Volume (V) with boundary (S), divisible into control volumes (Δ_V).

Engineers use integral identities based on two main ideas to solve electromagnetic field problems. These are Stokes' theorem for open areas $S_{1,2,3}$ bounded by a curve f and Gauss's divergence theorem for closed areas S. We demonstrate this in (24).

As many researchers have discussed in their works, such as in Refs. [13,14], the focus has been on resolving issues that are not stationary. In this context, Modifying the electromagnetic field equations into a form suited for enhancement, theoretical analysis, and computational techniques is appropriate. This change can be achieved by using Gauss's divergence theorem and the symmetry of the Kronecker delta operator (δ_{ij}, δ_{ji}). This modification applies to all time-dependent terms in the electromagnetic field equations, both for dielectric and conducting materials, where j and r are not equal to zero. The change, which will be explained in more detail below, makes the non-stationary term easier to use with Gauss's divergence theorem [42]:

$$\begin{cases} \overrightarrow{rotE} - \dfrac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \overset{\cdots}{\int_f} Edf + \overset{\cdots}{\int_f} \dfrac{\partial B}{\partial t} ndS_{1,2,3} = 0 \\ \dfrac{\partial B_i}{\partial t} = x_i \cdot \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial B_i}{\partial t}\right) + \dfrac{\partial B_i}{\partial t} \cdot \dfrac{\partial x_i}{\partial x_j} = \dfrac{\partial}{\partial x_j}\left[\dfrac{\partial B_i}{\partial t} x_i\right] \end{cases} \tag{32}$$

By applying Gauss's divergence theorem, the equations are further transformed into a format that facilitates analysis. Using relationship (32) as a foundation, new integral identities can be derived. These identities are now expressed over the closed surface, which incorporates all boundary conditions. As a result, we can formulate the following two equations:

$$\frac{\partial B_i}{\partial t} . x_i = \frac{\partial}{\partial x_j}\left[\frac{\partial B_i}{\partial t} x_i . x_i\right] \tag{33}$$

$$\overset{\cdots}{\int_S} (n \text{ X } E) ds + \overset{\cdots}{\int_S} \left(\frac{\partial B}{\partial t} n\right) . xds = 0 \tag{34}$$

In the next section, we'll show how to prove the previous equations by modifying Maxwell's laws to analyze the impact of variable terms in the field V on the frontier of the region S. The solution relies on symmetry and Gauss's divergence theorem, applied to both vector and scalar forms in analysis, computational techniques, and optimization, considering both conductive and insulating environments. The final section explores a model describing the interaction between ferrofluid and a field of magnets, applying Gauss's theorem to adjust the Navier-Stokes model [43].

4. Symmetry principles solve equations in non-conductive mediums

Symmetry plays a crucial role as a fundamental tool in various disciplines, particularly in electromagnetism and technical sciences [2]. In the field of electromagnetism, methods based on symmetry enable diverse approaches to studying orbital angular momentum and pseudo momentum. These approaches include contemporary techniques that incorporate spin-orbit coupling and explore electromagnetic knots. In technical sciences, symmetry is essential for maintaining system stability and helps in differentiating between symmetric and antisymmetric components within electromagnetism [15,37], in the field of symmetric electromagnetism, the following expressions can be written:

$$\frac{\partial B_i}{\partial x_j} = \frac{1}{2} \cdot \left(\frac{\partial B_i}{\partial x_j} + \frac{\partial B_j}{\partial x_i}\right) + \frac{1}{2} \cdot \left(\frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i}\right) \tag{35}$$

In (10) and (25), we use Gauss's theorem to rewrite Maxwell's equations and come up with a new way to understand them that ties symmetry to magnetic induction.

$$\left\| \begin{matrix} -\dfrac{\partial}{\partial t} & \varepsilon_{ijk}\dfrac{\partial}{\partial x_j} \\ \dfrac{\partial}{\partial x_i} & 0 \end{matrix} \right\| \cdot \begin{pmatrix} B_i \\ E_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{36}$$

On the left-hand side of Eq. (36), just the antisymmetric component (dE_k/dx_j) appears. We will now introduce a new derivative to Eq. (36) through an adjustment to its right-hand side. which contains a variable element. This modification can be written as follows:

$$\begin{cases} \dfrac{\partial B_j}{\partial x_j} = 0 \\ \overset{\cdots}{\int_V} \dfrac{\partial B_j}{\partial x_j} x_i dV = \overset{\cdots}{\int_S} B_j x_i . njdS - \overset{\cdots}{\int_V} B_j \dfrac{\partial x_i}{\partial x_j} dV = 0 \end{cases} \tag{37}$$

This adjustment allows us to reformulate the relation in a way that better reflects the system's antisymmetric nature. The previously mentioned relationship relies on three-dimensional space to perform integration on each of its terms. This approach allows us to analyze the phenomenon within a more comprehensive geometric framework, providing a deeper understanding of the relevant physical processes. And after applying the Kronecker delta symmetry modification ($\frac{\partial x_i}{\partial x_j} = A_{ij} = A_{ji}$), we obtain a modified form of the equation.

Furthermore, taking into account the assumption that $\frac{\partial}{\partial x_j}\left(\frac{\partial B_j}{\partial t}\right) = 0$, we can rewrite Eq. (38) in several different forms, each highlighting a specific aspect of the fundamental physical relationship:

$$\begin{cases} \overset{\cdots}{\int_V} B_i dV = \overset{\cdots}{\int_S} B_j x_i . nj . dS \overset{\text{Or in terms of vectors}}{\Rightarrow} \\ \qquad \overset{\cdots}{\int_V} B . dV = \overset{\cdots}{\int_S} (B . n) . x . dS \quad (a) \\ \overset{\cdots}{\int_V} \dfrac{\partial B_i}{\partial t} dV = \overset{\cdots}{\int_S} \dfrac{\partial B_j}{\partial t} x_i . njdS \overset{\text{Or in terms of vectors}}{\Rightarrow} \\ \qquad \overset{\cdots}{\int_V} \dfrac{\partial B}{\partial t} dV = \overset{\cdots}{\int_S} \left(\dfrac{\partial B}{\partial t} . n\right) . xdS \; (b) \end{cases} \tag{38}$$

The various forms of Eq. (38) allow for a deeper analysis and a wider understanding of the physical phenomenon under study.

Equation (38b) shows that the edges of a region can have unsteady magnetic flux density states, which could change the control volume method. We modify the structure of Eq. (36) for symmetry by employing Expression (38a) and the divergence theorem (Eq. (39)). This provides us with a novel formulation of Maxwell's equations [19,32].

$$\frac{\partial B_i}{\partial t} = \frac{\partial}{\partial x_j}\left[\frac{\partial B_j}{\partial t} . x_i\right] \tag{39}$$

$$\frac{\partial}{\partial x_j}\left[\varepsilon_{ijk}.E_K+\frac{\partial B_j}{\partial t}x_i\right]=0 \quad (40)$$

The first term in brackets in Eq. (40), which stands for the second-order antisymmetric tensor, can be broken down into symmetric and antisymmetric parts [12]. The existing approach incorporates the non-stationary component derived from average values:

$$\int_V \frac{\partial B}{\partial t}d(\Delta V)=\frac{\partial B_c}{\partial t}.\Delta V \quad (41)$$

$$\Delta V=\frac{1}{3}*\int_{\Delta S}(x.n_i).d(\Delta s) \quad (42)$$

Whereas the new method relies on Relation (40) for direct integration (Eq. (43)), with the same principle expressed in vector form (Eq. (44)).

$$\int_{\Delta V}\frac{\partial B_i}{\partial t}d(\Delta V)=\int_{\Delta V}\frac{\partial}{\partial x_j}\left(\frac{\partial B_j}{\partial t}x_i\right)d(\Delta V)=\int_{\Delta S}\left(\frac{\partial B_j}{\partial t}n_i\right).x_i d(\Delta s) \quad (43)$$

$$\int_{\Delta V}\frac{\partial B}{\partial t}d(\Delta V)=\int_{\Delta S}\left(\frac{\partial B}{\partial t}.n\right).xd(\Delta S)=0 \quad (44)$$

$$\int_S\left[(nXE)+\left(\frac{\partial B}{\partial t}.n\right).x\right]dS=0 \quad (45)$$

Eq. (45) is the integrated vector form of (40), showing that magnetic induction changes in$V$ originate at the system boundary. This aids in optimizing non-stationary tasks via boundary conditions. Later sections derive results for non-conductive spaces.

As widely acknowledged by researchers in the field of electricity, both charge density and conductivity are zero in non-conductive materials (ρe = σ = 0). In this situation, Maxwell's equations take on a symbolic form [44,45]:

$$\begin{gathered}\left\|\begin{matrix}-\frac{\partial}{\partial t} & \varepsilon_{ijk}\frac{\partial}{\partial x_j}\\ \frac{\partial}{\partial x_i} & 0\end{matrix}\right\|.\begin{pmatrix}D_i\\ H_k\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\ \text{(a)}\\ \left\|\begin{matrix}\frac{\partial}{\partial t} & \varepsilon_{ijk}\frac{\partial}{\partial x_j}\\ \frac{\partial}{\partial x_i} & 0\end{matrix}\right\|.\begin{pmatrix}B_i\\ E_k\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\ \text{(b)}\end{gathered} \quad (46)$$

By applying transform (39) to Equation (39a), Maxwell's equations can be represented in alternative forms, as demonstrated below:

$$If(39b)\Rightarrow^{then}\ddot{\int_V}\frac{\partial D}{\partial t}dV=\ddot{\int_S}\left(\frac{\partial D}{\partial t}.n\right).xdS \quad (47)$$

$$If(40)\Rightarrow^{then}\frac{\partial D_i}{\partial t}=\frac{\partial}{\partial x_j}\left[\frac{\partial D_j}{\partial t}.x_i\right] \quad (48)$$

$$If(41)\Rightarrow^{then}\frac{\partial}{\partial x_j}\left[\frac{\partial D_j}{\partial t}x_i-\varepsilon_{ijk}.H_K\right]=0 \quad (49)$$

$$If(46)\Rightarrow^{then}\int_S\left[\left(\frac{\partial D}{\partial t}.n\right)x-(n\times H)\right]dS=0 \quad (50)$$

The prior discussion clearly shows that we can restructure Maxwell's equations in a non-conductive environment. The conclusions derived in this context, including those pertaining to the control volumes method, remain relevant.

5. Scalar formulations of Maxwell's laws for insulating environments

We get the scalar form of the electromagnetic formula by multiplying Eq. (36) by the position vector x [18]. This makes it easier to analyze and improve the system's performance under different conditions.

$$\begin{cases} rotE.x=-\frac{\partial B}{\partial t}.x\\ \varepsilon_{ijk}.\frac{\partial E_k}{\partial x_j}.x_i=-\frac{\partial B_i}{\partial t}.x_i\end{cases} \quad (51)$$

$$divB=0\rightarrow\frac{\partial E_j}{\partial x_j}.x_ix_i=0 \quad (52)$$

$$\ddot{\int_V}\frac{\partial B_j}{\partial x_j}x_ix_idV=\ddot{\int_S}B_jx_ix_i.njdS-\ 2\ddot{\int_V}B_j\frac{\partial x_i}{\partial x_j}x_idV=0 \quad (53)$$

Through the last equation can be written:

$$\begin{cases}\ddot{\int_V}B_j\delta_{ij}x_idV=\frac{1}{2}\ddot{\int_S}B_jx_ix_in_jdS\\ \quad\text{in the vector form}\\ \ddot{\int_V}BxdV=\frac{1}{2}\ddot{\int_S}(B.n)(x.x)dS\end{cases} \quad (54)$$

Taking into account the divergence theorem in the form equals zero ($div\frac{\partial B}{\partial t}=0$), in this case, Eqs. (53) and (54) can be written as follows:

$$\ddot{\int_V}\frac{\partial B_i}{\partial t}x_idV=\frac{1}{2}\ddot{\int_S}\frac{\partial B_j}{\partial t}x_ix_in_jdS \quad (55a)$$

in vector notation

$$\ddot{\int_V}\frac{\partial B}{\partial t}xdV=\frac{1}{2}\ddot{\int_S}\left(\frac{\partial B}{\partial t}.n\right)\left(|x|^2\right)dS \quad (55b)$$

Leveraging the divergence theorem and applying it to (55a) yields a crucial relationship. This relationship provides the foundation for a novel reformulation of Maxwell's Eq. (51), offering a fresh perspective on electromagnetic theory (56a) and Substituting (56b) into (51) results in:

$$\frac{\partial B_i}{\partial t}x_i=\frac{1}{2}.\frac{\partial}{\partial x_j}\left(\frac{\partial B_j}{\partial t}x_ix_i\right) \quad (56a)$$

$$\frac{\partial}{\partial x_j}(\varepsilon_{ijk}.E_k+\frac{1}{2}\frac{\partial B_i}{\partial t}x_ix_i=0 \quad (56b)$$

The expression within the brackets can be decomposed into symmetric and antisymmetric components, mirroring the approach used in (40). When expressed in its integral form, this yields:

$$\begin{cases}\ddot{\int_S}\left[\frac{1}{2}.\frac{\partial B_j}{\partial t}x_ix_i+\varepsilon_{ijk}.x_i\right]n_jdS=0\\ \quad n\text{ the vector form}\\ \ddot{\int_S}\left[\frac{\partial B}{\partial t}.n|x|^2+2x(n\times E)\right]dS=0\end{cases} \quad (57)$$

A scalar formulation of Maxwell's laws, was derived under the assumption that Gauss's divergence theorem is valid. These relationships facilitate the assessment of numerical outcomes and indicate non-fixed variations in the induction of magnetism at regional boundaries, enabling the determination of time-dependent values. The equations are advantageous for optimization because of their scalar characteristics. Upon reevaluating the scenario of a non-conductive medium, Eq. (46) can be articulated in both its differential and integral representations, as demonstrated in Table 1.

Both analytical and numerical methods benefit from the improved version of Maxwell's equations.

6. Magnetic fields affect non-conductive magnetic liquids

In Sections 4 and 5, the authors present a new version of Maxwell's equations for non-conductive environments [5]. Using control volumes and the Maxwell stress tensor, they study incompressible fluid motion under non-conductive magnetic fields. The main goal is to develop a mathematical model of liquid flow in these conditions, requiring two key elements:

Ø The tensor of irreversible stresses in the liquid
Ø The volumetric magnetic force that the magnetic field applies to the liquid

The general relationships for both the electromagnetic force (F) and magnetic flux density (B_k) are given as follows [2,3,5,35]:

$$\begin{cases} F = \dfrac{\partial H_k}{\partial x_i} M_k \\ B_k = \mu_0 H_k + M_k \\ M_k = \chi H_k \end{cases} \tag{64}$$

In this context, p_e represents charge density for a non-conductive magnetic liquid ($p_e = 0$), B_k denotes magnetic flux density, H_k signifies magnetic field intensity, M_k indicates magnetization, and χ represents magnetic susceptibility.

The Kelvin force, or magnetic force density, is given by:

$$\begin{cases} F = \dfrac{\chi}{2} \dfrac{\partial H^2}{\partial x_i} \\ H^2 = H_i.H_i \\ M^2 = M_i.M_i \end{cases} \tag{65}$$

Table 1
Differential and integral forms of maxwell's equations: original and new variants for non-conductive media.

Differential form	
Original equations	**New variant**
$\left\Vert \begin{matrix} -\dfrac{\partial}{\partial t} & \varepsilon_{ijk}\dfrac{\partial}{\partial x_j} \\ \dfrac{\partial}{\partial x_i} & 0 \end{matrix} \right\Vert \begin{pmatrix} D_i \\ H_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (46a)	$\dfrac{\partial}{\partial x_j}\left[\dfrac{\partial D_j}{\partial t} x_i x_i - 2\varepsilon_{ijk}\dfrac{\partial H_k}{\partial x_j}.x_i\right] = 0$ (58) $\dfrac{\partial}{\partial x_j}\left[\dfrac{\partial B_i}{\partial t} x_i x_i + 2\varepsilon_{ijk}\dfrac{\partial E_k}{\partial x_j}.x_i\right] = 0$ (59)
$\left\Vert \begin{matrix} \dfrac{\partial}{\partial t} & \varepsilon_{ijk}\dfrac{\partial}{\partial x_j} \\ \dfrac{\partial}{\partial x_i} & 0 \end{matrix} \right\Vert \begin{pmatrix} B_i \\ E_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (46b)	
Integral form	
Original equations	**New variant**
$\iint_V \dfrac{\partial D}{\partial t} x dV = \iint_S (nXH).xdS$ (60)	$\iint_S \left(\dfrac{\partial D}{\partial t}.n\right) \lvert x \rvert^2 dS = 2\iint_S (n \times H) x dS$ (62)
$\iint_V \dfrac{\partial B}{\partial t} x dV = -\iint_S (nXE).xdS$ (61)	$\iint_S \left(\dfrac{\partial B}{\partial t}.n\right) \lvert x \rvert^2 dS = -2\iint_S (n \times E) x dS$ (63)

Nonstationary variables $D_{x,t}$ and $B_{x,t}$ originates around the system frontier, affecting their behavior within volume V. The scalar Maxwell equations, which come from Gauss's theorem, make numerical analysis and optimization easier by linking changes in induction to boundary conditions and variables that change over time.

Magnetic liquid flow models use the Maxwell stress tensor, considering non-Newtonian behavior and including force$F$ [6,46]:

$$\begin{cases} F = \dfrac{\partial M_{ij}}{\partial x_j} \\ M_{ij} = \dfrac{\chi}{2} \delta_{ij} H_k.H_k \end{cases} \tag{66}$$

Magnetic liquid equilibrium equations are:

$$\begin{cases} \dfrac{\partial}{\partial x_j}\left[\rho\left(\dfrac{\partial v_j}{\partial t} - g_j\right) x_i + \rho v_i v_j - \delta_{ij} - M_{ij}\right] = 0 \\ \delta_{ij} = -\rho\delta_{ij} + \delta_{ij} \\ g_i = \dfrac{\partial}{\partial x_i}(g_k x_k) \end{cases} \tag{67}$$

For an incompressible magnetic liquid, the continuity equation is $\left(\frac{\partial v_i}{\partial x_i} = divv = 0\right)$. Using Eq. (40), we can reformulate the Navier-Stokes equation for this scenario:

$$\frac{\partial}{\partial x_j}\left[\rho.\frac{\partial v_j}{\partial t} x_i + \rho v_i v_j - \delta_{ij} - \rho\delta_{ij}\left(g_k x_k - \frac{1}{2}\chi H^2\right)\right] = 0 \tag{68}$$

Applying the Divergence theorem [2–6,35], we can express this equation in a new integral form:

$$\oint_S \left[\rho\left(\frac{\partial v}{\partial t}.n\right).x + \rho(v.n).v - \sigma - \left(gx - \frac{1}{2}\chi(H.H)\right).n\right] dS = 0 \tag{69}$$

The advantages of the new variant are evident when comparing Eqs. (66) with (67) in numerical solutions using the finite volume method and boundary condition analysis. Assuming ($divA = 0$), the following holds for (38b):

$$\iint_V \frac{\partial v}{\partial t} dV = \iint_S \left(\frac{\partial v}{\partial t}.n\right).xdS \tag{70}$$

Accordingly, the continuity equation results in:

$$\iint_S \frac{\partial v}{\partial t} n.dS = 0 \ and \ \iint_S \frac{\partial B}{\partial t}.ndS \tag{71}$$

What is significant about these relations is that solely the normal component of $(\partial B/\partial t)$ manifests the influence of not stationary elements $\frac{\partial B}{\partial t}.n$ in the area V on the system's limits.

Gauss's divergence theorem is used to develop new formulations of Maxwell's eqs. for non-stationary magnetic fluids. This approach is significantly simpler than classical methods, which rely on complex differential forms that are difficult to handle numerically. By converting the eqs. into integral forms, Gauss's theorem emphasizes boundary effects, simplifying analysis and simulation. The method is highly effective for studying interactions between electromagnetic fields and magnetic fluids in dynamic conditions. It supports applications such as ferrofluid-based cooling, plasma generators, and magnetic dampers. For applied researchers, it offers precise tools for analyzing complex electromagnetic systems and improving the design of medical devices, precision motors, and renewable energy systems affected by time-varying fields by linking internal behavior to boundary conditions.

7. Conclusions

In this work, we have developed a new formulation of the not stationary Maxwell's equations based on Gauss's divergence theorem and applied this approach to magnetic fluid dynamics. The study demonstrates that the new model provides a more accurate representation of the interaction between electromagnetic fields and unsteady magnetic flow, further enhancing our understanding of these systems' behavior under nonlinear and complex conditions. The modified equations exhibit remarkable flexibility in incorporating additional effects, such as viscosity and magnetic conductivity, within a coherent mathematical framework.

Moreover, this formulation opens promising prospects for future research, particularly in the fields of numerical modeling and spectral analysis of complex dynamic systems. This work can be expanded to include more intricate thermal and temporal effects or be integrated with models of smart and phase-changing materials, such as ferromagnetic materials and magnetic nanofluids. It is also expected that this theoretical framework will contribute significantly to improving the design of microelectromechanical systems (MEMS) and advancing magnetic energy storage technologies.

Finally, the findings of this study represent a significant step towards the development of advanced analytical tools for studying electromagnetic phenomena in heterogeneous media, paving the way for numerous applications in applied physics and energy engineering, as well as magnetic resonance techniques and 3D printing technologies based on magnetic fields.

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Data availability statement

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

CRediT authorship contribution statement

Mohammed Bouzidi: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Abdelfatah NASRI:** Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Mohamed Ben Rahmoune:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Oussama Hafsi:** Writing – original draft, Visualization, Validation, Resources, Methodology, Investigation, Data curation, Conceptualization. **Dessalegn Bitew Aeggegn:** Writing – review & editing, Visualization, Validation, Resources, Project administration, Investigation, Conceptualization. **Sherif S. M. Ghoneim:** Writing – review & editing, Visualization, Validation, Supervision, Resources, Project administration, Funding acquisition, Conceptualization. **Enas Ali:** Writing – review & editing, Visualization, Validation, Resources, Investigation, Conceptualization. **Ramy N. R. Ghaly:** Writing – review & editing, Visualization, Validation, Resources, Investigation, Conceptualization.

Declaration of competing interest

Authors stated that no conflict of Interest.

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Data availability

Data will be made available on request.

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